Lecture 18
14.7 Second Derivative Test and Extreme Value Theorem

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Things to note

Upcoming dates:
Today: WF Drop Date
Wednesday: Review
Friday: Exam 2

Last class

1. $f(a,b)$ is a <u>local maximum</u> of f if $f(a,b) \geq f(x,y)$ for all points (x,y) in the domain of f near (a,b).
2. $f(a,b)$ is a <u>local minimum</u> of f if $f(a,b) \leq f(x,y)$ for all points (x,y) in the domain of f near (a,b).

Theorem

If $f(x,y)$ has a local min or max at (a,b) and $f_x(a,b)$, $f_y(a,b)$ are defined, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$. Another way to say this is $\nabla f(a,b) = \vec{\mathbf{0}}$.

Definition

$f(x,y)$ has a <u>saddle point</u> at a critical point (a,b) if (a,b) isn't a local max and (a,b) isn't a local min.

The second derivative test

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Definition

The discriminant (or Hessian) of a function $f(x, y)$ is the function

$$H(f) = f_{xx}f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}.$$

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Suppose that $f, f_x, f_y, f_{xx}, f_{yy}$ are continuous near (a, b) with $f_y(a, b) = f_x(a, b) = 0$. The following hold:

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Theorem

Suppose that $f, f_x, f_y, f_{xx}, f_{yy}$ *are continuous near* (a, b) *with* $f_y(a, b) = f_x(a, b) = 0$. *The following hold:*

i) If $f_{xx}(a, b) < 0$ *and* $H(f) > 0$ *at* (a, b), *then* f *has a local max at* (a, b).

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i) If $f_{xx}(a, b) < 0$ and $H(f) > 0$ at (a, b), then f has a local max at (a, b).

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iii) If $H(f) < 0$ at (a, b), then f has a saddle point at (a, b).

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Suppose that $f, f_x, f_y, f_{xx}, f_{yy}$ are continuous near (a, b) with $f_y(a, b) = f_x(a, b) = 0$. The following hold:

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iii) If $H(f) < 0$ at (a, b), then f has a saddle point at (a, b).

iv) The test is inconclusive at (a, b) if $H(f) = 0$ at (a, b).

Example

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Let $f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$. *Classify all critical values of* f.

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We have $f_x(x, y) = y - 2x - 2$ and $f_y(x, y) = x - 2y - 2$. To find critical values, we have to solve the system

$$y - 2x - 2 = 0 \text{ and } x - 2y - 2 = 0.$$

This has the solution $(-2, -2)$.

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This has the solution $(-2,-2)$.
To determine whether $(-2,-2)$ is an extreme value, we have to determine whether $H(f)$ is positive or negative at $(-2,-2)$. We have $f_{xx}(x,y) = -2$, $f_{yy}(x,y) = -2$, and $f_{xy}(x,y) = 1$. So in this case $H(f)$ is constant and equals $(-2)(-2) - 1^2 = 3 > 0$. So $(-2,-2)$ is a local max or a local min. Then because $f_{xx}(-2,-2) < 0$, we have $(-2,-2)$ is a local max.

Another example

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We find $f_x(x, y) = -6x + 6y$ and $f_y(x, y) = 6y - 6y^2 + 6x$. Setting the first equal to zero gives $x = y$, and substituting this into the second equation gives $12x - 6x^2 = 6x(2 - x) = 0$, so $x = 0$ or $x = 2$. Thus the critical points are $(0, 0)$ and $(2, 2)$.

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$$H(f) = (-6)(6 - 12y) - 6^2 = -36 + 72y - 36 = 72(y - 1).$$

At $(0, 0)$, $H(f)$ is $72(0 - 1) < 0$, meaning $(0, 0)$ is a saddle point. At $(2, 2)$, $H(f)$ is $72(2 - 1) > 0$ and $f_{xx}(2, 2) = -6 < 0$, so $(2, 2)$ is a local max.

Absolute Extrema

We need some definitions to talk about extrema that are *absolute* rather than just local.

Definition

Let R be a region in the plane. We say

R is <u>bounded</u> if it lies inside a disk of finite radius.

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We will think of closed and bounded regions as the two-dimensional version of closed intervals.
These definitions will be necessary for the extreme value theorem and will also be useful in Chapter 15 for integration.

Extreme value theorem

Theorem

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This is reminiscent of the extreme value theorem for single-variable functions.

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1. List all interior points of R where there is a local max/min/saddle point.
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This process breaks our search down into two cases: points (a, b) in the interior of R and points (a, b) on the boundary of R.

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Find the absolute max/min of $f(x, y) = 2 + 2x + 4y - x^2 - y^2$ *on the triangular region bounded by* $x = 0$, $y = 0$, *and* $y = 9 - x$.

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1. List all interior points of R where there is a local max/min/saddle point.

$f_x(x, y) = 2 - 2x$ and $f_y(x, y) = 4 - 2y$. Setting these equal to 0, we get $x = 1$ and $y = 2$. This gives the point $(1, 2)$, which is in the interior of R.

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On the boundary line $y = 0$, we have $f(x, 0) = 2 + 2x - x^2$. So $f'(x) = 2 - 2x$, and setting this equal to 0 gives $x = 1$. Thus the point $(1, 0)$ is a possible extrema on R. We also take the end points $(0, 0)$ and $(9, 0)$.

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On the boundary line $y = 0$, we have $f(x,0) = 2 + 2x - x^2$. So $f'(x) = 2 - 2x$, and setting this equal to 0 gives $x = 1$. Thus the point $(1,0)$ is a possible extrema on R. We also take the end points $(0,0)$ and $(9,0)$.
On the boundary line $x = 0$, we have $f(0,y) = 2 + 4y - y^2$, so $f'(y) = 4 - 2y$ and setting this equal to 0 gives $y = 2$. So $(0,2)$ and the end point $(0,9)$ are the possible extrema from this portion of the boundary.

2. continued

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On the boundary line $y = 9 - x$, we have

$$f(x, 9-x) = 2 + 2x + 4(9-x) - x^2 - (9-x)^2 = -2x^2 + 16x - 43.$$

Thus $f'(x) = -4x + 16$. Setting this equal to 0 gives $x = 4$. Substituting into $y = 9 - x$, we find $y = 5$. So we add the point $(4, 5)$ to our search.

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$f(1,2) = 7$
$f(0,0) = 2$
$f(1,0) = 3$

$f(9,0) = -61$
$f(0,2) = 6$
$f(0,9) = -43$

$f(4,5) = -11$

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$f(1,2) = 7$	$f(9,0) = -61$	$f(4,5) = -11$
$f(0,0) = 2$	$f(0,2) = 6$	
$f(1,0) = 3$	$f(0,9) = -43$	

Thus the absolute maximum of f on R is 7 and occurs at the local max $(1,2)$. The absolute minimum of f on R is -61 and occurs on the boundary point $(9,0)$.